On a singular point in the Newtonian theory of hypersonic flow

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An examination of the Newtonian inviscid theory of hypersonic flow past a sphere is made in the neighbourhood of the singular point which occurs at $\theta = 60^\circ$. A uniformly valid theory is developed in this neighbourhood which exhibits the limiting detached 'free-layer' behaviour postulated by Hayes and Lighthill in the limit as $\gamma \to 1$. A complete solution is obtained which gives details of streamline and shock wave shape, pressure distribution, etc. for $\gamma - 1$ small.

Reasons are given why the empirical 'modified' Newtonian theory of Lees proves to be a good approximation to experimentally determined pressure distributions.

A novel check to the theory is provided by a simple power-law relation between the pressure on the sphere surface and the distance of the shock wave away from sphere at the same point.

1. Introduction

The Newtonian theory of hypersonic flow has been developed in the past few years from the original ideas of Busemann (1933). A review of this work is given by Hayes & Probstein (1959), to which the reader is referred for numerous references. A solution to the problem of flow of a hypersonic inviscid stream past a bluff body is achieved by an expansion procedure in powers of ρ_0/ρ_s, where ρ is the density and the suffices 0 and s denote values in the free stream and behind the bow shock wave, respectively. The free-stream Mach number is generally assumed infinite although this is not essential (e.g. Chester 1956). In the case of a perfect gas, the above ratio is simply $(\gamma-1)/(\gamma+1)$, where γ is the ratio of specific heats, and the expansion is then carried out in powers of $\epsilon = (\gamma-1)/(\gamma+1)$. No rigorous proof has been given that such an expansion will converge and close examination of the first few terms would seem to indicate that it is, at best, only slowly convergent. However, the solution obtained is the correct solution in the limit as $\epsilon \to 0$. For this reason, one might expect that it would give results which bear a resemblance to those for $\epsilon \neq 0$; but both numerical computations using the complete equations and experimental results would seem to indicate that this is not so. In particular, the pressure predicted on the surface of the body seems to bear little relationship to that observed in experiments. Lees (1955) has suggested that the correct form for reduction of experimental results on a sphere is given by

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$C_p/C_{p_{max}} = \cos^2\theta$, where C_p is the pressure coefficient, $C_{p_{max}}$ its value at the stagnation point and θ is the angle measured from the stagnation point. This result would seem to indicate that, to obtain a theoretical justification, one need simply admit that the pressure is determined by a spherical shock wave with a suitable adjustment for the pressure at the stagnation point. Although this result has the virtue of simplicity, it adds little to our understanding of the theoretical problem. Moreover, the theoretical result as suggested by Busemann (1933) would require a more complicated expression for the pressure distribution, since the pressure is determined not only by the shock-pressure rise, but also by the centrifugal effect due to the curvature of the body. The resulting pressure distribution is then made up of a pressure rise at the shock wave followed by a pressure drop between the shock wave and body. The resulting form of the pressure variation on a sphere in the limit $\epsilon \to 0$ is then

$$C_p/C_{p_{max}} = \sin 3\theta/3\sin\theta$$

or, alternatively,

$$C_p/C_{p_{max}} = \cos^2\theta - \tfrac{1}{3}\sin^2\theta.$$

The final term is the contribution from the centrifugal pressure gradient between shock wave and body. It should be noticed that the pressure becomes zero for $\theta = 60°$ when the first and second terms above become equal. At this point, the assumptions of the Busemann theory are no longer valid and the solutions obtained from the theory have a singular behaviour. The author believes that it is at this point that the answer to the discrepancy between theory and experiment may be found. The non-uniformity of the convergence of the solution as $\epsilon \to 0$ near this point produces a singular behaviour which influences a large part of the flow field.

In this paper, we shall investigate the nature of the singularity near this point and postulate a solution which converges uniformly in this neighbourhood. Although it is impossible to state whether this solution is unique, it would seem to possess all the characteristics required of it. The problem, as it presents itself, is as follows: For $\theta < 60°$, we have a solution as determined by the Newtonian theory in which the pressure tends to zero like $(\frac{1}{3}\pi - \theta)$ and the distance of the shock wave away from the sphere (measured radially) becomes infinite like $(\frac{1}{3}\pi - \theta)^{-\frac{2}{3}}$. For $\theta > 60°$, it was originally suggested by Busemann that the flow 'separates'. This idea has led Lighthill (1957) and Hayes & Probstein (1959) to postulate that the fluid is flung away from the sphere at this point and proceeds as a 'free-layer', the pressure on the body surface becoming identically zero. If the assumption is then made that the fluid proceeds in a thin layer close to the shock wave, it is possible to obtain an analytic expression for the shape of the shock wave. The 'free-layer' solution of Lighthill (1957) and Hayes & Probstein (1959) has the limiting form for the shock shape

$$\tfrac{1}{6}(\theta - \tfrac{1}{3}\pi)^3$$

near $\theta = \frac{1}{3}\pi$. For the Newtonian solution, the shock layer thickness is proportional to ϵ, while for the free-layer solution it is independent of ϵ. Thus, we are required to find a solution to the problem which behaves like

$$\epsilon(\tfrac{1}{3}\pi - \theta)^{-\frac{2}{3}} \quad \text{for} \quad \theta < \tfrac{1}{3}\pi, \tag{1}$$

and like

$$(\theta - \tfrac{1}{3}\pi)^3 \quad \text{for} \quad \theta > \tfrac{1}{3}\pi. \tag{2}$$

The type of solution required is shown schematically in figure 1. If we consider an expanded system of co-ordinates and an expanded shock shape represented by a relationship of the form

$$\epsilon^{\alpha} Y[(\theta-\tfrac{1}{3}\pi)\,\epsilon^{\beta}], \tag{3}$$

these conditions require that

$$Y(z) \to z^3 \quad \text{as} \quad z \to +\infty \tag{4}$$

and

$$Y(z) \to (-z)^{-\frac{2}{3}} \quad \text{as} \quad z \to -\infty. \tag{5}$$

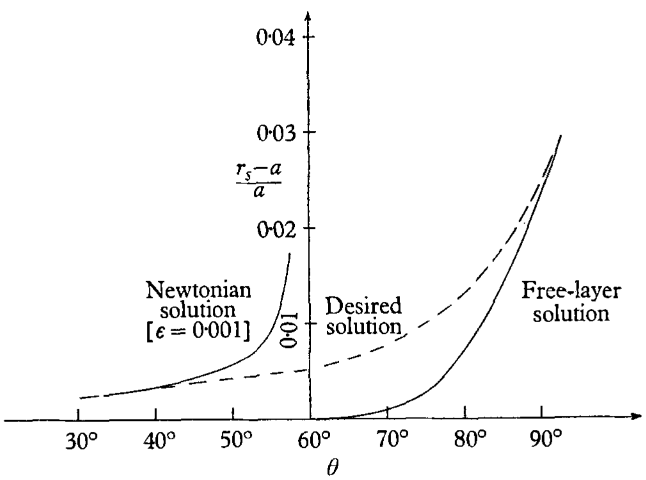

FIGURE 1. The desired form of solution for the shock shape.

In order that the dependence on ϵ be correct, we therefore require

$$\left.\begin{aligned} \alpha + 3\beta &= 0, \\ \alpha - (\tfrac{2}{3})\beta &= 1, \end{aligned}\right\} \tag{6}$$

by equating exponents of ϵ. Thus

$$\beta = -\tfrac{3}{11} \quad \text{and} \quad \alpha = \tfrac{9}{11}. \tag{7}$$

The shock shape is, therefore, given in the form

$$\epsilon^{\frac{9}{11}} Y(\{\theta - \tfrac{1}{3}\pi\}\,\epsilon^{-\frac{3}{11}}). \tag{8}$$

The limiting forms of the solution for Y are given by (4) and (5). It will be shown below that Y satisfies a non-linear second-order differential equation. It does not seem possible to obtain an analytic expression for Y. However, the equation can be integrated numerically and then we see from (8) that the solution is known for all ϵ. By the type of argument used above, it is possible to deduce the required behaviour of the shock shape with ϵ. It is possible to determine much more from the complete theory. The behaviour of the whole flow field can be determined in the neighbourhood of the singular point. The procedure used is to expand the variables, both dependent and independent, in series with coefficients dependent upon powers of ϵ, in a similar fashion to that suggested by Lighthill (1949). In fact, we will only consider the first term in this expansion. This we will do mainly to reduce the complexity of the problem. It should also be necessary to examine the expansions at other points in the flow field if the higher order terms were required.

Most of the results obtained below were determined by the author in a previous paper (1958). The method used, however, relied primarily on arguments on the physical mechanism of the transition from a Newtonian theory to a detached free-layer solution. In that paper, it was argued that the most important effect to be taken into account at the singular point is that of curvature of streamlines relative to the body. This result is confirmed in the present paper. The fact that this is so is perhaps the most encouraging result of the work. It will be realized that the result of Lees (1955) already indicates that this might be the case. For if the pressure drop due to body curvature is to be nullified, some mechanism is required which will give an increased surface pressure. The most obvious one is that the streamline curvature relative to the body should become important. In fact, the theory shows that this curvature is of the same order as the body curvature at the singular point. As, in practice, the singular point behaviour influences the whole flow field (due to the fact that ϵ is not mathematically very small), the experimental predictions become more understandable.

The most disconcerting feature of the theory is, however, already obvious from the form of (8). It will be seen that near the singular point the dependence is on $\epsilon^{\frac{3}{11}}$ rather than on ϵ. A more detailed study of the flow field shows that this is, indeed, the case. In practice, $\epsilon^{\frac{3}{11}}$ is never small enough for the theory to hold. In other words, the transition from the Newtonian to the free-layer solution depends on $\epsilon^{\frac{3}{11}}$ being small and the free-layer solution will only be achieved when this condition is satisfied.

The theory has been developed below for the case of a sphere; but the theory is not restricted only to the sphere. Similar results could be obtained for any two-dimensional or axially symmetric body. The dependence on ϵ would, however, be different although it would seem straightforward by arguments similar to those used above to determine the correct form.

More generally, the approach to the problem is that used by the author (1956) in his treatment of Newtonian theory. The shock layer is considered to be a narrow region in which rates of changes across the layer are much larger than those along it. By consideration of the orders of magnitude of the variables, it is then possible to simplify the equations of motion considerably. In the region near the singular point, however, the orders of magnitude of the variables change and we are required to modify the equations accordingly.

2. The equations of motion

By considering a particular body shape (the sphere) we are able to choose a co-ordinate system immediately simplifying the analysis considerably. We shall choose a spherical polar system of co-ordinates with r and θ measured from the centre of the sphere and the forward stagnation point, respectively, $r = a$ is the surface of the sphere.

The equations of continuity, momentum and energy can then be written in the form

$$\frac{\partial}{\partial r}(\rho u r^2 \sin\theta) + \frac{\partial}{\partial \theta}(\rho v r \sin\theta) = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{2.2}$$

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = 0, \tag{2.3}$$

$$u\frac{\partial i}{\partial r} + \frac{v}{r}\frac{\partial i}{\partial \theta} = \frac{1}{\rho}\left[u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \theta}\right], \tag{2.4}$$

where u, v are the velocities in the r and θ directions; ρ is the density; p the pressure; and i the enthalpy. If we assume a perfect gas with constant specific heats then

$$i = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}. \tag{2.5}$$

Introduction of the Stokes' stream function, ψ, to satisfy equation (2.1), then gives the following set of equations:

$$\frac{\partial \psi}{\partial r} = \rho v r \sin\theta, \quad \frac{\partial \psi}{\partial \theta} = -\rho u r^2 \sin\theta, \tag{2.6}$$

$$\frac{\partial u}{\partial \theta} - v + \frac{\partial p}{\partial \psi} r^2 \sin\theta = 0, \tag{2.7}$$

$$v\frac{\partial v}{\partial \theta} + uv + \frac{1}{\rho}\left\{\frac{\partial p}{\partial \theta} - \frac{\partial p}{\partial \psi}\rho u r^2 \sin\theta\right\} = 0, \tag{2.8}$$

$$\frac{\partial i}{\partial \theta} = \frac{1}{\rho}\frac{\partial p}{\partial \theta}, \tag{2.9}$$

where θ and ψ are taken as the independent variables in equations (2.7) to (2.9).

The Newtonian theory may then be deduced by considering a new system of variables

$$\left.\begin{aligned} \rho' &= \epsilon\frac{\rho}{\rho_\infty}, \quad p' = \frac{p}{\rho_\infty U_\infty^2}, \quad v' = \frac{v}{U_\infty}, \quad u' = \frac{u}{\epsilon U_\infty}, \\ \psi' &= \frac{\psi}{\rho_\infty U_\infty a^2}, \quad r' = \frac{r-a}{a\epsilon}, \quad \theta' = \theta. \end{aligned}\right\} \tag{2.10}$$

All the dashed variables are then assumed of order unity. The equations (2.7) to (2.9) then reduce to

$$\frac{\partial p}{\partial \psi} = \frac{v}{a^2 \sin\theta}, \quad \frac{\partial v}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial i}{\partial \theta} = 0 \tag{2.11}$$

which together with (2.6) then give the Newtonian approximation. The shock layer is assumed thin and, hence, approximately spherical. Equations (2.11) can then be integrated to give

$$p = \frac{1}{a^2 \sin\theta}\int_{\psi_s}^{\psi} v(\psi)\, d\psi + p_s,$$

where s denotes values at the shock wave. Using the strong shock wave approximations we have

$$p = \left\{\frac{\sin 3\theta + \sin^3 \xi}{3 \sin \theta}\right\} \rho_\infty U_\infty^2, \tag{2.12}$$

where

$$\psi = \tfrac{1}{2}\rho_\infty U_\infty a^2 \sin^2 \xi.$$

Equation (2.6) then gives

$$\frac{r-a}{a} = 3\epsilon \int_0^{\sin \xi} \frac{(1-s^2)\,ds}{\sin 3\theta + s^3}. \tag{2.13}$$

This is then the equation of the streamlines, $\xi =$ const., and $\xi = \theta$ is the equation of the shock wave.

	Order of magnitude	
Variable	Newtonian	Near singular point
$p/\rho_\infty U_\infty^2$	1	$\epsilon^{\frac{3}{11}}$
u/U_∞	ϵ	$\epsilon^{\frac{7}{11}}$
v/U_∞	1	$\epsilon^{\frac{1}{11}}$
ρ/ρ_∞	ϵ^{-1}	$\epsilon^{-\frac{8}{11}}$
$\psi/a^2\rho_\infty U_\infty$	1	$\epsilon^{\frac{2}{11}}$
$(r-a)/a$	ϵ	$\epsilon^{\frac{9}{11}}$
θ	1	—
$\frac{1}{3}\pi - \theta$	—	$\epsilon^{\frac{3}{11}}$

TABLE 1

From equation (2.13), it is clear that at $\theta = \frac{1}{3}\pi$, we have some kind of singular behaviour. In fact, we have

$$\frac{r-a}{a} \sim \frac{2\pi}{3^{\frac{7}{6}}} \left(\tfrac{1}{3}\pi - \theta\right)^{-\frac{2}{3}}, \tag{2.14}$$

provided that

$$\lim_{\theta \to \frac{1}{3}\pi} \left[\sin \xi / (\sin 3\theta)^{\frac{1}{3}}\right] \to \infty.$$

Similarly, the pressure on the sphere surface ($\xi = 0$) tends to zero like

$$(2/\sqrt{3})\left(\tfrac{1}{3}\pi - \theta\right). \tag{2.15}$$

The breakdown of the theory is thus associated with the vanishingly small pressure at the sphere surface. When this occurs, it is no longer possible to assume orders of magnitude for the variables as in equation (2.10). Is it possible, though, to determine orders of magnitude for the variables in the neighbourhood of the singular point and, hence, obtain a uniformly valid solution in this region? In § 1, an argument was put forward which gives a uniformly valid form near $\theta = 60°$. This result indicates that near this point, the shock layer has a thickness which is order $a\epsilon^{\frac{9}{11}}$ and the rates of change along the shock layer are order $a\epsilon^{\frac{3}{11}}$. A close examination of the equations (2.1) to (2.4) then leads to the orders of magnitude shown in table 1, where they are also compared with the original orders of magnitude of the Newtonian theory. We therefore consider a new system of variables such that

$$\left.\begin{aligned} p &= \epsilon^{\frac{3}{11}}\rho_\infty U_\infty^2 P(z,\zeta), \\ u &= \epsilon^{\frac{7}{11}} U_\infty U(z,\zeta), \\ v &= \epsilon^{\frac{1}{11}} U_\infty V(z,\zeta), \\ \rho &= \epsilon^{-\frac{8}{11}}\rho_\infty \Omega(z,\zeta), \\ \psi &= \tfrac{1}{2}a^2\rho_\infty U_\infty \epsilon^{\frac{2}{11}}\zeta^2, \\ r-a &= \epsilon^{\frac{9}{11}} a R(z,\zeta), \end{aligned}\right\} \tag{2.16}$$

where $$\theta - \tfrac{1}{3}\pi = \epsilon^{\frac{3}{11}} z \quad \text{and} \quad \sin\xi = \epsilon^{\frac{1}{11}}\zeta. \tag{2.17}$$

Substituting in equations (2.6) to (2.8), we obtain

$$\zeta\frac{\partial\zeta}{\partial R} = \Omega V\frac{\sqrt{3}}{2}, \quad \zeta\frac{\partial\zeta}{\partial z} = -\Omega U\frac{\sqrt{3}}{2} \tag{2.18}$$

$$\frac{\partial V}{\partial z} = O(\epsilon^{\frac{9}{11}}) \quad \text{and} \quad \frac{\partial}{\partial z}\left(\frac{P}{\Omega}\right) = O(\epsilon),$$

where terms of order $\epsilon^{\frac{3}{11}}$ have been neglected in the first two equations. These equations therefore remain essentially the same as for the 'Newtonian' theory. At first sight, it would appear that equation (2.9) does also, since this becomes

$$\frac{1}{\zeta}\frac{\partial P}{\partial\zeta}\frac{\sqrt{3}}{2} - V\left(1 - \frac{z}{\sqrt{3}}\epsilon^{\frac{3}{11}}\right) + \epsilon^{\frac{3}{11}}\frac{\partial U}{\partial z} = 0. \tag{2.19}$$

However, the shock layer in the expanded co-ordinate system now extends from $\zeta = 0$ to $\zeta \sim \epsilon^{-\frac{1}{11}}$ and, hence, it is possible that the terms in $\epsilon^{\frac{3}{11}}$ can contribute.

Integrating equation (2.19), we obtain

$$P = \frac{2}{\sqrt{3}}\int_{\zeta_s}^{\zeta}\left\{V - \epsilon^{\frac{3}{11}}\left[\frac{Vz}{\sqrt{3}} + \frac{\partial U}{\partial z}\right]\right\}\zeta\, d\zeta + P_s, \tag{2.20}$$

where suffix s denotes values at the shock wave.

Now on the shock wave $V = U_\infty\cos\phi$, where Φ is the angle the shock wave makes with the free stream. Now $\phi = \frac{1}{2}\pi - \theta + O(\epsilon^{\frac{6}{11}})$ and, hence, $V = U_\infty\{\sin\theta + O(\epsilon^{\frac{6}{11}})\}$ on the shock wave. Also, since the value of ψ on the shock wave is related only to the slope of the shock wave, we have $\theta = \xi + O(\epsilon^{\frac{6}{11}})$ at the shock wave. Hence, using (2.16) and (2.18), we obtain

$$V = \zeta + O(\epsilon^{\frac{5}{11}}). \tag{2.21}$$

Similarly
$$\begin{aligned} p_s &= \rho_\infty U_\infty^2(1-\epsilon)\sin^2\phi \\ &= \rho_\infty U_\infty^2\{\cos^2\theta + O(\epsilon^{\frac{6}{11}})\}, \end{aligned}$$

and thus
$$P = \tfrac{1}{4}\epsilon^{-\frac{3}{11}} - \frac{\sqrt{3}}{2}z + O(\epsilon^{\frac{3}{11}}). \tag{2.22}$$

Again
$$\begin{aligned} \zeta = \zeta_s &= \epsilon^{\frac{1}{11}}\sin\theta \\ &= \epsilon^{-\frac{1}{11}}\frac{\sqrt{3}}{2}\left(1 + \frac{\epsilon^{\frac{3}{11}}}{\sqrt{3}}z\right) \end{aligned} \tag{2.23}$$

on the shock wave.

Substituting in equation (2.20), we have

$$P = \tfrac{1}{4}\epsilon^{-\frac{3}{11}} - \frac{\sqrt{3}}{2}z - \frac{2}{\sqrt{3}}\int_{\zeta}^{\zeta_s}\zeta^2\,d\zeta + \frac{2}{\sqrt{3}}\epsilon^{\frac{3}{11}}\int_{\zeta}^{\zeta_s}\zeta^2\,d\zeta + \frac{2}{\sqrt{3}}\epsilon^{\frac{1}{11}}\int_{\zeta}^{\zeta_s}\frac{\partial U}{\partial z}\zeta\,d\zeta + O(\epsilon^{\frac{3}{11}})$$

$$= -\frac{2}{\sqrt{3}}z + \frac{2}{\sqrt{3}}\frac{\zeta^3}{3} + \frac{2}{\sqrt{3}}\epsilon^{\frac{3}{11}}\int_{\zeta}^{\sqrt{\frac{3}{2}}\epsilon^{-1/11}}\frac{\partial U}{\partial z}\zeta\,d\zeta + O(\epsilon^{\frac{3}{11}})$$

$$= -\frac{2}{\sqrt{3}}z + \frac{2}{\sqrt{3}}\frac{\zeta^2}{3} + \frac{2}{\sqrt{3}}\int_{\zeta}^{\sqrt{\frac{3}{2}}\epsilon^{-1/11}}\frac{\partial}{\partial z}\left(\frac{U}{V}\right)\zeta^2\,d\zeta + O(\epsilon^{\frac{3}{11}})$$

$$= -\frac{2}{\sqrt{3}}z + \frac{2}{3\sqrt{3}}\zeta^3 + \frac{1}{4}\left\{\frac{\partial}{\partial z}\left(\frac{U}{V}\right)\right\}_{\zeta=\infty} + o(\epsilon^{\frac{3}{11}}). \tag{2.24}$$

Now, from equation (2.18), we have

$$\frac{U}{V} = -\frac{\partial\zeta/\partial z}{\partial\zeta/\partial R} = \left(\frac{\partial R}{\partial z}\right)_{\zeta} \tag{2.25}$$

and, hence,
$$P = -\frac{2}{\sqrt{3}}z + \frac{2}{3\sqrt{3}}\zeta^3 + \frac{1}{4}\left(\frac{\partial^2 R}{\partial z^2}\right)_{\zeta=\infty} + o(\epsilon^{\frac{3}{11}}). \tag{2.26}$$

Thus, the pressure is modified from the Newtonian value which comprises the first two terms by a term which depends on the curvature of the streamlines at $\zeta = \infty$. In the expanded system of co-ordinates, $\zeta = \infty$ corresponds to the shock wave.

The distance of the streamlines away from the body is obtained from equations (2.18) as

$$R = \frac{2}{\sqrt{3}}\int_0^{\zeta}\frac{d\zeta}{\Omega}. \tag{2.27}$$

Since
$$\frac{P}{\Omega} = \frac{P_s}{\Omega_s} = 1 + O(\epsilon^{\frac{3}{11}}), \tag{2.28}$$

we obtain
$$R = \frac{2}{\sqrt{3}}\int_0^{\zeta}\frac{d\zeta}{P} = \frac{2}{\sqrt{3}}\int_0^{\zeta}\frac{d\zeta}{\frac{\sqrt{3}}{4}(-z) + \frac{2}{3\sqrt{3}}\zeta^3 + \frac{1}{4}\left(\frac{\partial^2 R}{\partial z^2}\right)_{\zeta=\infty}}. \tag{2.29}$$

If we let $R_1 = R(z, \infty)$, then

$$R_1 = \frac{2^{\frac{2}{3}}}{\left\{\frac{2}{\sqrt{3}}(-z) + \frac{1}{4}\frac{d^2 R_1}{dz^2}\right\}^{\frac{2}{3}}}\int_0^{\infty}\frac{d\eta}{1+\eta^3} = \frac{8\pi}{3\sqrt{3}}\left\{\frac{d^2 R_1}{dz^2} + \frac{8}{\sqrt{3}}(-z)\right\}^{-\frac{2}{3}}.$$

The equation of the shock wave is then given by

$$\frac{d^2 R_1}{dz^2} - \frac{8}{\sqrt{3}}z = \left(\frac{8\pi}{3\sqrt{3}}\right)^{\frac{3}{2}} R_1^{-\frac{3}{2}}. \tag{2.30}$$

Near the sphere itself, where ζ is still small, the equations of the streamlines are given by

$$R = \frac{2}{\sqrt{3}} \int_0^{\zeta} \frac{d\zeta}{\frac{2}{\sqrt{3}}(-z) + \frac{1}{4}\frac{d^2R_1}{dz^2} + \frac{2}{3\sqrt{3}}\zeta^2}, \tag{2.31}$$

where R_1 is calculated from equation (2.30).

The pressure is obtained from equation (2.26); and, in particular, the pressure on the surface of the sphere is

$$P = \frac{2}{\sqrt{3}}(-z) + \frac{1}{4}\frac{d^2R_1}{dz^2}. \tag{2.32}$$

We have thus reduced the problem to solving equation (2.30). It does not seem possible to do this analytically; but since equation (2.30) only need be solved to give a complete solution, the numerical evaluation can be done once and for all. The boundary condition on equation (2.30) is given at $z = -\infty$, where R_1 must behave like the original Newtonian solution. The details of the types of solution of equation (2.30) and the method of solution are given in the Appendix.

In the original co-ordinate system, the shock wave shape is expressible in the form

$$\frac{r-a}{a} = \epsilon^{\frac{9}{11}} R_1[\epsilon^{-\frac{3}{11}}(\theta - \tfrac{1}{3}\pi)], \tag{2.33}$$

where R_1 is the solution of (2.30).

The surface pressure is given in the form

$$p = \rho_\infty U_\infty^2 \epsilon^{\frac{3}{11}} P_b[\epsilon^{-\frac{3}{11}}(\theta - \tfrac{1}{3}\pi)], \tag{2.34}$$

where $$P_b(z) = \frac{2}{\sqrt{3}}(-z) + \frac{1}{4}\frac{d^2R_1}{dz^2},$$

and the streamlines by

$$\frac{r-a}{a} = \epsilon^{\frac{9}{11}} G[\epsilon^{-\frac{3}{11}}(\theta - \tfrac{1}{3}\pi), \epsilon^{-\frac{1}{11}} \sin \xi], \tag{2.35}$$

where $$G(z, \zeta) = \frac{3}{F(z)^{\frac{2}{3}}} \int_0^{\zeta/[\{F(z)\}^{\frac{1}{3}}]} \frac{d\zeta}{1+\zeta^3}$$

and $$F(z) = \frac{3\sqrt{3}}{2} P_b(z).$$

We see, therefore, that in the limit as $\epsilon \to 0$, the shock wave approaches the limiting forms of the solution at $\pm\infty$ depending on whether $\theta \gtrless \frac{1}{3}\pi$. The solution to equation (2.30) is such that

$$R_1 \sim (-z)^{-\frac{2}{3}} \quad \text{as} \quad z \to -\infty$$
$$\sim z^3 \quad \text{as} \quad z \to +\infty.$$

In the limit of $\epsilon \to 0$, we have

$$\frac{r-a}{a} \sim \epsilon\,(\tfrac{1}{3}\pi - \theta)^{-\frac{2}{3}} \quad \text{for} \quad \theta < \tfrac{1}{3}\pi$$

and $$\frac{r-a}{a} \sim (\theta - \tfrac{1}{3}\pi)^3 \quad \text{for} \quad \theta > \tfrac{1}{3}\pi.$$

We can also investigate the behaviour of the streamlines as $\epsilon \to 0$. For $\zeta = \infty$ the streamlines very close to the shock wave are identical with it. However when ζ is not large the same limiting behaviour occurs when

$$\frac{2^{\frac{1}{3}} \sin \xi}{\sqrt{3}\, \epsilon^{\frac{1}{11}} \{P_b[\epsilon^{-\frac{3}{11}}(\theta - \frac{1}{3}\pi)]\}^{\frac{1}{3}}} \to \infty. \tag{2.36}$$

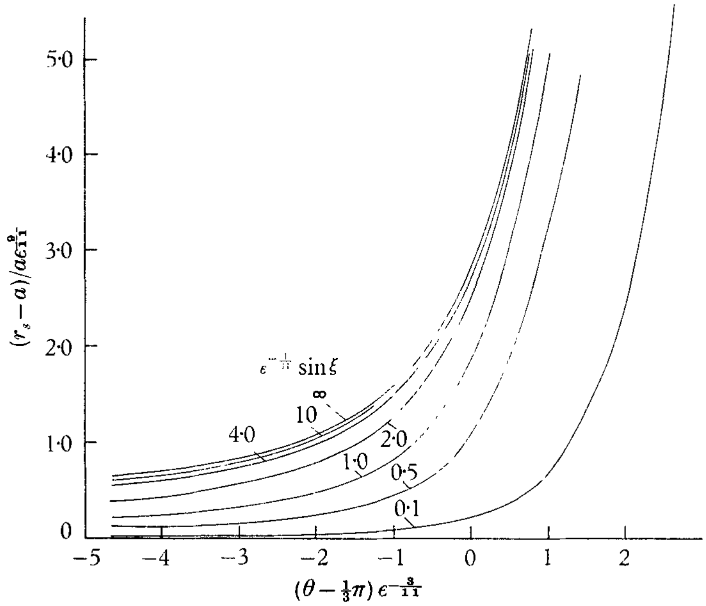

FIGURE 2. The shock shape ($\epsilon^{-\frac{1}{11}} \sin \xi = \infty$) and the streamlines ($\epsilon^{-\frac{1}{11}} \sin \xi =$ const.) in the neighbourhood of the singular point ($\theta = 60°$).

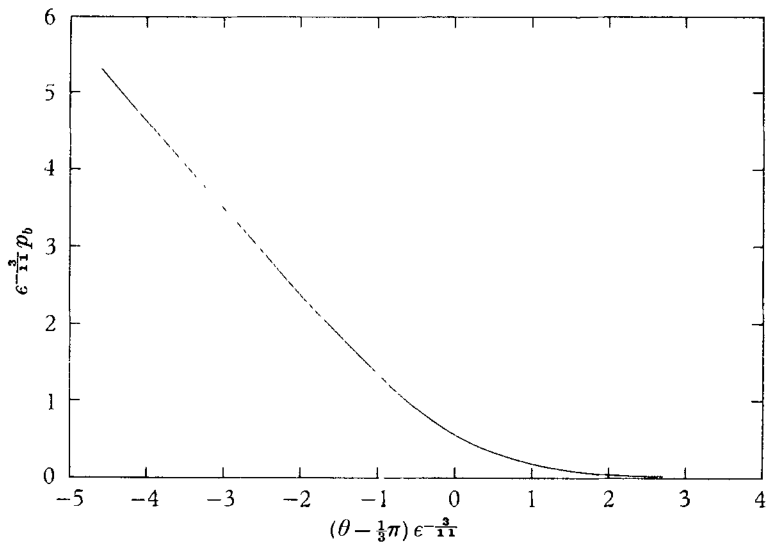

FIGURE 3. Pressure variation at the surface of the sphere in the neighbourhood of the singular point $\theta = 60°$.

For $\theta < \frac{1}{3}\pi$, this occurs when
$$\frac{\sin\xi}{(\frac{1}{3}\pi-\theta)^{\frac{1}{3}}} \to \infty,$$
and for $\theta > \frac{1}{3}\pi$, when
$$\epsilon^{-\frac{1}{2}}(\theta-\tfrac{1}{3}\pi)^{\frac{3}{2}}\sin\xi \to \infty.$$

Hence, for $\theta < \frac{1}{3}\pi$, the streamlines are distributed independently of ϵ, whereas for $\theta > \frac{1}{3}\pi$, the streamlines approach the shock wave at a rate proportional to $\epsilon^{-\frac{1}{2}}$ as ϵ tends to zero. Thus, we see from the equation for the shock wave shape that the region of transition from the Newtonian theory to the free-layer solution is of order $\epsilon^{\frac{3}{11}}$, but after separation, the free-layer solution is achieved comparatively rapidly like $\epsilon^{-\frac{1}{2}}$. The pressure along the body surface falls rapidly to zero like
$$\epsilon^{\frac{3}{2}}(\theta-\tfrac{1}{3}\pi)^{-\frac{9}{2}}.$$

The solution to equation (2.30) was calculated using the method outlined in the Appendix. The results are plotted in figure 2, together with the equations of the streamlines computed from equation (2.35). The rapidity of the transition to the free-layer solution is immediately evident although the scale of phenomena is proportional to $\epsilon^{\frac{3}{11}}$. The surface pressure is computed from equation (2.34) and is plotted in figure 3. In each case, the expanded co-ordinates are used to give universal curves for all ϵ.

3. Physical implications of the theory

As mentioned in § 1, it has been difficult in the past to try to compare directly the experimental and theoretical results, since the discrepancy between the two was inconsistent with the original Newtonian theory. It is clear, however, from the theory of § 2 that the modification necessary to correct the pressure distribution given by Newtonian theory is to take into account the curvature of the streamlines relative to the body. To a first approximation, this curvature may be taken equal to the curvature of the shock wave. This prediction may be checked in a rather interesting manner, for using (2.32) and (2.30) we see that
$$\frac{p_b}{\rho_\infty U_\infty^2} \propto \frac{\epsilon^{\frac{3}{2}}}{[(r_s-a)/a]^{\frac{3}{2}}}. \tag{3.1}$$
The pressure on the surface of the sphere p_b is, therefore, related to the distance of the shock from the sphere by an inverse three-halves power law. This result is plotted in figure 4 for two sets of experimental results: the first in air at $M_\infty = 5{\cdot}8$, due to Oliver (1956), and the second in helium at $M_\infty = 14{\cdot}0$, due to Vas, Bogdonoff & Hammitt (1958).

Direct comparison of the results with experiment is difficult, however, since ϵ is never small enough physically for the theory to hold. The transition region, discussed in the previous section, in practice spans the region from stagnation point to the shoulder of the sphere. Thus, the theory shows that the singular behaviour of the Newtonian result will tend in practice to influence the whole flow field. Comparison of the pressure with experimental results shows that the theoretical predictions are considerably larger than the experimental values for $\epsilon = \frac{1}{6}$ and $\frac{1}{4}$. This is due to the singular behaviour influencing the whole region including the stagnation region. Since the pressure is in the first approximation a linear function away from the singular point (as shown by the first term of (2.32)), this would tend to over-emphasize the pressure in the stagnation region, where the Newtonian theory shows a nearly constant value. It should be noted, however, that equation (2.30) cannot be used right up to the stagnation point. For, it is clear that the factor multiplying d^2R/dz^2 in (2.30) is the momentum flow in the shock layer at $\theta = 60°$. It would be more correct, therefore, to reduce this as the

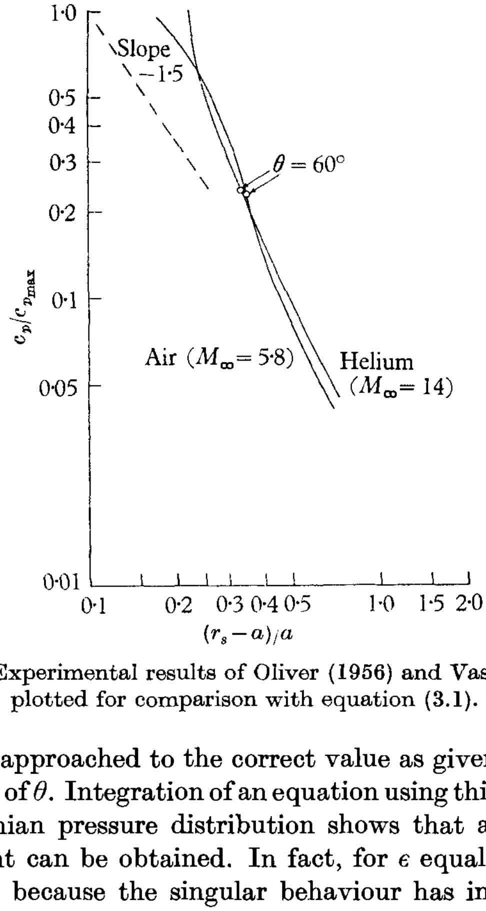

FIGURE 4. Experimental results of Oliver (1956) and Vas *et al.* (1958) plotted for comparison with equation (3.1).

stagnation point is approached to the correct value as given by the Newtonian theory at that value of θ. Integration of an equation using this correction and also the correct Newtonian pressure distribution shows that a distribution much closer to experiment can be obtained. In fact, for ϵ equal to $\frac{1}{6}$ or $\frac{1}{4}$ a further complication arises because the singular behaviour has influenced this linear variation even at the stagnation point. This effect is rather disconcerting, since the result obtained by using a much cruder equation for the pressure based on physical reasoning similar to the above gave a much closer agreement with experiment (cf. Freeman 1958). Of course, it is now clear that, mathematically, these additional refinements are unjustified within the uniformly valid approximation.

4. Conclusion

The theory outlined above shows the nature of the solution to the hypersonic inviscid flow past a sphere in the neighbourhood of the singular point obtained when $\epsilon \to 0$ for free-stream Mach number infinite. The result also indicates that the curvature of the streamlines relative to the body is the most important factor in this region. Although in practice ϵ is never so small that this part of the flow field is independent of the other parts, the limiting form of the solution is of interest in showing how the transition from the initially attached Newtonian flow to the detached free-layer solution would occur.

It is clear from the results that the free-layer solution is reached fairly rapidly after the transition region (figure 2).

It would seem possible to develop similar theories for other bluff bodies, both two-dimensional and axially symmetric. The method would be essentially the same. A general formulation of the problem for any bluff body would seem to be rather complicated, however.

Perhaps a more interesting development from the above theory could be obtained by considering other Newtonian solutions and analyzing the solution in the neighbourhood of discontinuities of slope, curvature, etc. In this way, it would seem possible to develop a uniformly valid solution in the neighbourhood of such points.

It seems that the procedure developed in this paper is an extremely powerful one in obtaining solutions to the hypersonic bluff body problem. If one considers that the result obtained by the author in his previous paper (1956) to be the fundamental Newtonian solution, then the theory of the type developed above may be considered as the solution to a particular problem in a region where that solution does not have uniform validity. In a similar way, it is possible to obtain other Newtonian solutions by simply requiring that in other limiting processes a uniformly valid solution is required. The author has succeeded in obtaining the solution for a blunt body originally derived by Hayes (see Hayes & Probstein 1959, Chap. 5) and Serbin (1956).

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Appendix

Solution to equation (2.30)

We require a solution to the equation

$$\frac{d^2R_1}{dz^2}-\frac{8}{\sqrt{3}}z = \left(\frac{8\pi}{3\sqrt{3}}\right)^{\frac{3}{2}} R_1^{-\frac{3}{2}}, \tag{A 1}$$

subject to the boundary condition that at $z = -\infty$ the solution is obtained by neglecting the d^2R_1/dz^2 term.

A change of variables

$$y = \frac{3^{\frac{23}{22}}}{8^{\frac{5}{11}}\pi^{\frac{9}{11}}} R_1,$$

and

$$x = \frac{3^{\frac{2}{11}}2^{\frac{28}{33}}}{\pi^{\frac{3}{11}}} z \tag{A 2}$$

reduces the equation to the form

$$\frac{d^2y}{dx^2} = x+\frac{1}{y^{\frac{3}{2}}}. \tag{A 3}$$

It does not seem possible to get an analytic solution to (A 3) and hence we must resort to numerical methods. The behaviour of the equation (A 3) near the point

$x = -\infty$ is somewhat complicated as can be seen from perturbing the known asymptotic form. For, putting

$$y = (-x)^{-\frac{2}{3}} + y_1,$$

we obtain

$$\frac{d^2y_1}{dx^2} + \tfrac{3}{2}(-x)^{\frac{5}{3}}\, y_1 = -\tfrac{10}{9}(-x)^{-\frac{8}{3}} \qquad \text{(A 4)}$$

neglecting the higher order terms. We see that for x near $-\infty$ the above equation has complementary functions of the form

$$y_1 \sim \frac{1}{(-x)^{\frac{5}{12}}} \exp\left\{\pm i \tfrac{6}{11} \sqrt{\tfrac{3}{2}}\, (-x)^{\frac{11}{6}}\right\} \qquad \text{(A 5)}$$

which tend to zero in an oscillatory manner with x but whose derivatives become infinite. The particular integral of (A 4) is the solution we require. It is clear, therefore, that we must use some procedure to get away from $x = -\infty$ before we use the normal integration procedure. We do this by using the asymptotic solution of (A 3) which is

$$y = (-x)^{-\frac{2}{3}} \sum_{n=0}^{\infty} b_n(-x)^{-\frac{11}{3}n}, \qquad \text{(A 6)}$$

where b_0, b_1, b_2, ... are given in Table 2.

n	b_n
0	1·0
1	0·740741
2	12·0988
3	−602·554
4	60,999·4

TABLE 2

This solution is only slowly convergent but it was found possible to use the above expansion up to $x = -6$. The integration was then continued using the usual integration procedure up to $x = +6$, taking intervals of 0·25.

An asymptotic expansion can also be obtained for large x and fitted to the solution obtained by numerical integration.

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